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On conservation laws in elastodynamics

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Abstract

The objective of this investigation is the establishment of governing balance and conservation laws in elastodynamics. The feature of the approach employed here consists in placing time on the same level as the space coordinates, as is done in the theory of relativity, i.e., pursuing a 4×4 formalism. Both the Lagrangian and the Eulerian descriptions of the postulated Lagrangian function are formulated. The Euler–Lagrange equations in each of the two descriptions are discussed, as well as the results of the application of the gradient, divergence and curl. The latter two operations are made to act on the product of coordinates and the Lagrangian function, i.e., a four-vector. In this manner a variety of balance and conservation laws are derived, partly well known and partly seemingly novel. In each case the general results for elastodynamics are illustrated for the simple case of a linearly elastic bar.

Keywords: Elastodynamics; Balance law; Conservation law; 4×4 Formalism; Lagrangian description; Eulerian description; Elastic bar

1. Introduction

Several methods exist in the theory of fields to establish the governing conservation and balance laws. The procedure used probably most widely is that of Noether (1918), based on her first theorem including an extension by Bessel-Hagen (1921). The starting point in this procedure is a Lagrangian function. If such a function for a system under consideration does not exist, due, for instance, to the presence of some form of dissipation, then a relatively recently established, so-called neutral-action method (cf. Honein et al., 1991; Kienzler and Herrmann, 2000) can be used. It can be shown that for systems which do possess a Langrangian function, the neutral-action method leads to the same results as the Noether procedure, including the Bessel-Hagen extension. In addition to the above two methods, a third procedure may be employed by means of direct submission of the Lagrangian (for fields for which it exists) to the differential operators of *grad*, *div* and *curl*, in symbolic notation ∇, $\nabla\bullet$ and $\nabla\times$, respectively. The latter two operations are applied to a vector $\mathbf{x}L$ where $\mathbf{x}$ are the independent coordinates and L is the Lagrangian function.

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The procedure has been successfully used in elastostatics in material space, leading to the path-independent integrals commonly known as **J**, M and **L**. It is obvious that the operators *grad*, *div* and *curl* are related to the transformation of translation, self-similarity and rotation, respectively.

The authors have become interested recently in a 4×4 formalism in discussing the basic laws of continuum mechanics because, as they found, it permits to reveal hidden relationships between, e.g., conservation of mass and balance of linear physical momentum (stress) (Herrmann and Kienzler, 1999; Kienzler and Herrmann, 2003).

Thus it appeared intriguing to investigate the results of the application of the three differential operators to the elastodynamic field considering the time not as a parameter, as is usual, but as an independent variable on the same level as the three space coordinates.

In the first section of this paper we lay out the required general formalism for an arbitrary system specified by a Lagrangian function with an unspecified number of dependent and independent variables.

In the following sections this formalism is applied to an elastodynamic system, first in the Lagrangian formulation in which the independent variables are the reference coordinates and time. To make certain that all four coordinates have the same dimension, the time is multiplied by some characteristic velocity, as is done in relativity theory where the characteristic velocity is the speed of light. As a result, three balance laws are derived and it is investigated under which conditions, if any, these balance laws reduce to conservation laws.

An analogous procedure is followed for the Eulerian formulation of elastodynamics in which the dependent and independent variables are interchanged leading to analogous results.

As a specific and most simple example a linearly elastic bar is considered in both formulations, leading to the necessity of defining some new quantities (such as for example a "kinetic force"), and to some seemingly novel balance and conservation laws.

It may be mentioned that conservation laws in general, and the newly derived conservation laws in particular, have a broad range of applicability. They are useful in studying problems of defect and fracture mechanics, stability of surfaces and interfaces, moving phase transformations, melting and mass-accretion, etc. They are also valuable tools in establishing global existence and uniqueness theorems and in the improvement of algorithms in numerical procedures (cf. Olver, 1993).

2. General system

2.1. Euler–Lagrange equations

We consider a system possessing a Lagrangian function L which depends on the independent variables ξ^μ, the dependent variables ϕ^m and the covariant derivatives $\phi^m{}_{,\nu}$. The range of μ and m shall be arbitrary

$$L = L(\xi^\mu; \phi^m, \phi^m{}_{,\nu}). \tag{1}$$

The system is embedded in a space with metric coefficients

$$g^{\mu\nu} = g^{\nu m}, \qquad g_{\mu\nu} = g_{\nu\mu}. \tag{2}$$

The associated Euler–Lagrange equations are

$$E_m(L) = \frac{\partial L}{\partial \phi^m} - \left(\frac{\partial L}{\partial \phi^m{}_{,\nu}} \right)_{,\nu} = 0. \tag{3}$$

We furthermore introduce the abbreviation

$$\left(\frac{\partial L}{\partial \xi^{\mu}}\right)\bigg|_{\text{expl}} = j_{\mu}. \tag{4}$$

2.2. Application of grad

We first evaluate the gradient of L, i.e.,

$$L_{,\mu} = \left(\frac{\partial L}{\partial \xi^{\mu}}\right)\bigg|_{\text{expl}} + \frac{\partial L}{\partial \phi^{m}}\frac{\partial \phi^{m}}{\partial \xi^{\mu}} + \frac{\partial L}{\partial (\phi^{m}{}_{,v})}\frac{\partial \phi^{m}{}_{,v}}{\partial \xi^{\mu}}. \tag{5}$$

We assume that the space under consideration is free of curvature and that, therefore, the order of the differentiation in the last term is interchangeable. We apply the rule for differentiation of products. Further, we express $L_{,\mu}$ as $\delta^{v}_{\mu}L_{,v}$ where δ^{v}_{μ} is the Kronecker delta. The result of these manipulation leads to

$$(L\delta^{v}_{\mu})_{,v} = j_{\mu} + \phi^{m}{}_{,\mu}\left[\frac{\partial L}{\partial \phi^{m}} - \left(\frac{\partial L}{\partial (\phi^{m}{}_{,v})}\right)_{,v}\right] + \left(\frac{\partial L}{\partial (\phi^{m}{}_{,v})}\phi^{m}{}_{,\mu}\right)_{,v} \tag{6}$$

which can be rewritten as

$$\left[L\delta^{v}_{\mu} - \frac{\partial L}{\partial \phi^{m}{}_{,v}}\phi^{m}{}_{,\mu}\right]_{,v} = j_{\mu} + E_{m}(L)\phi^{m}{}_{,\mu}. \tag{7}$$

It is seen that the tensor in brackets, which shall be labelled $T^{v}{}_{\mu}$, is divergence-free along solutions, i.e., $E_{m}(L) = 0$, and provided the Lagrangian does not depend explicitly on ξ^{μ}, i.e., $j_{\mu} = 0$,

$$T^{v}{}_{\mu,v} \equiv \left[L\delta^{v}_{\mu} - \frac{\partial L}{\partial \phi^{m}{}_{,v}}\phi^{m}{}_{,\mu}\right]_{,v} = 0. \tag{8}$$

In general field theories, $T^{v}{}_{\mu}$ is called the energy–momentum tensor.

2.3. Application of div

We next evaluate the divergence of the vector $\xi^{\mu}L$, i.e., $(\xi^{\mu}L)_{,\mu}$

$$(\xi^{\mu}L)_{,\mu} = \delta^{\mu}_{\mu}L + \xi^{\mu}L_{,\mu}. \tag{9}$$

We set $\delta^{\mu}_{\mu} = \alpha$ which indicates the dimensionality of the underlying space. The second term above shall be subjected to the same manipulations as in the preceding section, namely interchange of the order of differentiation and application of the rule for differentiation of products.

Further, similarly as before, we set

$$(\xi^{\mu}L)_{,\mu} = (\xi^{\mu}\delta^{v}_{\mu}L)_{,v}. \tag{10}$$

By substitution, the following results

$$\left[\xi^{\mu}T^{v}{}_{\mu} + \phi^{m}\frac{\partial L}{\partial (\phi^{m}{}_{,v})}\right]_{,v} = \alpha L + \frac{\partial L}{\partial \phi^{m}}\phi^{m} + j_{\mu}\xi^{\mu}, \tag{11}$$

where, as before $T^{v}{}_{\mu}$ is given by (8).

We introduce next the following definition: We shall call a system to be homogeneous of grade n, provided the relation holds

$$nL = \frac{\partial L}{\partial \phi^m{}_{,\nu}} \phi^m{}_{,\nu}. \tag{12}$$

For such a system it follows with (3)

$$\alpha L = \frac{\alpha}{n}\left[\left(\frac{\partial L}{\partial \phi^m{}_{,\nu}}\phi^m\right)_{,\nu} - \frac{\partial L}{\partial \phi^m}\phi^m\right], \tag{13}$$

and the substitution results in

$$\left[\xi^\mu T^\nu{}_\mu + \frac{n-\alpha}{n}\phi^m \frac{\partial L}{\partial(\phi^m{}_{,\nu})}\right]_{,\nu} = \frac{n-\alpha}{n}\frac{\partial L}{\partial \phi^m}\phi^m + j_\mu \xi^\mu. \tag{14}$$

It is seen that a conservation law exists only if L does depend neither on ξ^μ (explicitly) nor on ϕ^m. With these restrictions and the notation

$$V^\nu \equiv \xi^\mu T^\nu{}_\mu + \frac{n-\alpha}{n}\phi^m \frac{\partial L}{\partial(\phi^m{}_{,\nu})}, \tag{15}$$

where V^ν might be called the virial of the system, we have

$$V^\nu_{,\nu} = 0. \tag{16}$$

By further manipulations and provided $E_m(L) = 0$, i.e., along solutions, $V^\nu_{,\nu}$ can be expressed as

$$V^\nu_{,\nu} = \xi^\mu T^\nu{}_{\mu,\nu} = 0. \tag{17}$$

2.4. Application of curl

The rotation of a vector $\xi_\mu L$ in α-dimensional space may be represented as

$$\varepsilon^{\gamma\delta\cdots\mu\nu}(\xi_\mu L)_{,\nu}, \tag{18}$$

where $\varepsilon^{\gamma\delta\cdots\mu\nu}$ is the completely skew-symmetric permutation tensor of rank α. The rotation of a vector is a tensor of rank $\alpha - 2$.

The expression $(\xi_\mu L)_{,\nu}$ may be subjected to similar manipulations as in the preceding sections. In addition, the metric tensor is applied to lower or raise tensor indices, e.g.,

$$\xi_\mu = g_{\mu\lambda}\xi^\lambda, \tag{19}$$

$$T_{\mu\nu} = g_{\mu\lambda}T^\lambda{}_\nu. \tag{20}$$

The result of a straightforward manipulation is

$$\varepsilon^{\gamma\delta\cdots\mu\nu}[\xi_\mu T^\lambda{}_\nu]_{,\lambda} = \varepsilon^{\gamma\delta\cdots\mu\nu}(T_{\mu\nu} + \xi_\mu j_\nu), \tag{21}$$

where the energy–momentum tensor appears in a mixed form and in a purely covariant form, $T^\lambda{}_\nu$ and $T_{\mu\nu}$, respectively.

It is seen that a conservation law exists, provided $j_\nu = 0$ (independence of L on ξ^ν) and, in addition, $T_{\mu\nu}$ be symmetric, i.e.,

$$T_{\mu\nu} = T_{\nu\mu}, \tag{22}$$

and this conservation law has the form

$$\varepsilon^{\gamma\delta\cdots\mu\nu}(\xi_\mu T^\lambda{}_\nu)_{,\lambda} = 0. \tag{23}$$

3. Elastodynamics in Lagrangian description

3.1. Lagrangian function

We consider an elastic body in motion with mass density $\rho_o = \rho_o(X^J)$ in a reference configuration. We identify the independent variables ξ^μ as

$$\begin{aligned} \xi^0 &= \hat{t} = c_0 t, \\ \xi^J &= X^J, \quad J = 1,2,3. \end{aligned} \tag{24}$$

The independent variable $\hat{t}$ and not t is used in order for all independent variables to have the same dimensions, where c_o is some velocity. In the special theory of relativity, $c_o = c =$ velocity of light is used, to obtain a Lorentz-invariant formulation in Minkowski space. In the non-relativistic theory, an alternative invariant formulation does not exist and c_o may be chosen arbitrarily, e.g., as some characteristic wave speed or it may be normalized to one. The independent variables $\xi^J = X^J$ $(J = 1,2,3)$ are the space-coordinates in the reference configuration of the body.

Although $\xi^0 = \hat{t}$ and $\xi^J = X^J$ have the same dimensions, the time t is an independent variable and is not related to the space-coordinates by a proper time τ as in the theory of relativity. Therefore, we will deal with Galilean-invariant objects, four "vectors" and 4×4 matrices that will be called (cf. Morse and Feshbach, 1953) "tensors", although the formulation is not covariant. Special care has to be used when dealing with *div* and *curl*, where time- and space-coordinates are coupled.

The current coordinates are designated by x^i and play the role of the dependent variables or fields ϕ^m. They are defined by the mapping or *motion*

$$x^i = \chi^i(\hat{t}, X^J), \quad i = 1,2,3. \tag{25}$$

The derivatives are

$$\frac{\partial x^i}{\partial \xi^0} = \frac{\partial x^i}{c_o \partial t}\bigg|_{X^J\,\text{fixed}} = \frac{1}{c_o} v^i, \tag{26}$$

$$\frac{\partial x^i}{\partial X^J} = F^i{}_J \tag{27}$$

with the physical velocities v^i and the deformation gradient $F^i{}_J$ as the Jacobian of the mapping (25). For later use, the determinant of the Jacobian is introduced

$$J_F = \det[F^i{}_J] > 0. \tag{28}$$

The Lagrangian function that will be treated further is thus identified as

$$L = L(\hat{t}, X^J; x^i, v^i, F^i{}_J). \tag{29}$$

In more specific terms, we postulate the Lagrangian to be the kinetic potential

$$L = T - \rho_o(W + V) \tag{30}$$

with the densities of the

- kinetic energy $T = \rho_o(X^J)v^i v_i$,
- strain energy $\rho_o W = \rho_o W(\hat{t}, X^J; F^i{}_J)$, (31)
- force potential $\rho_o V = \rho_o V(\hat{t}, X^J; x^i)$.

The various derivatives of L allow their identifications in our specific case as indicated

$$\frac{\partial L}{\partial x^i} = \rho_o f_i \quad \text{Volume force,} \tag{32}$$

$$\frac{\partial L}{\partial v^j} = c_o \rho_o v_j \quad \text{Momentum,} \tag{33}$$

$$\frac{\partial L}{\partial (F^i{}_J)} = -P^J{}_i \quad \text{First Piola–Kirchhoff stress,} \tag{34}$$

$$\left.\frac{\partial L}{\partial \hat{t}}\right|_{\text{expl}} = j_0 \frac{1}{c_o} \quad \text{Energy source term,} \tag{35}$$

$$\left.\frac{\partial L}{\partial X^J}\right|_{\text{expl}} = j_J \quad \text{Inhomogeneity force.} \tag{36}$$

The Euler–Lagrange equations are in this case the equations of motion

$$\frac{\partial}{\partial t}(\rho_o v_j) - P^I_{j,I} - \rho_o f_j = 0. \tag{37}$$

3.2. Energy–momentum “tensor”

The “tensor” $T^\nu{}_\mu$ in the Lagrangian description is given as

$$T^\nu{}_\mu = L\delta^\nu{}_\mu - \frac{\partial L}{\partial (x^j{}_{,\nu})} x^j{}_{,\mu} \tag{38}$$

which is identical to the energy–momentum “tensor” discussed by Morse and Feshbach (1953). The various components of $T^\nu{}_\mu$ may be identified as follows

$$-T^0{}_0 = H = T + \rho_o(W + V) \quad \text{Hamiltonian (total energy),} \tag{39}$$

$$-T^0{}_J = \rho v_j F^j{}_J c_o = c_o R_J \quad \text{Field-(or wave-)momentum density,} \tag{40}$$

$$-T^I{}_0 = -P^I{}_j v^j \frac{1}{c_o} = -\frac{1}{c_o} S^I \quad \text{Field intensity, or energy flux,} \tag{41}$$

$$-T^I{}_J = (\rho_o(W + V) - T)\delta^I_J - P^I{}_j F^j{}_J = b^I{}_J \quad \text{Material momentum or Eshelby tensor.} \tag{42}$$

It is noted that if $j_0 = 0$, then $T^\nu{}_{0,\nu} = 0$, stating that energy is conserved. Further, if $j_J = 0$, i.e., the body is homogeneous, then $T^\nu{}_{J,\nu} = 0$, stating that the material momentum, involving the Eshelby tensor, in conserved. This is stated in explicit formulae

$$T^\nu{}_{0,\nu} = 0 = -\frac{1}{c_o}\left[\frac{\partial H}{\partial t} - \frac{\partial}{\partial X^J}(P^J{}_j v^j)\right], \tag{43}$$

$$T^{\nu}{}_{I,\nu} = 0 = -\left[\frac{\partial}{\partial t}(\rho_o v_j F^j{}_I) + \frac{\partial}{\partial X^J} b^J{}_I\right]. \tag{44}$$

3.3. Application of div and curl

If W is homogeneous of degree $n = 2$, i.e., for a linearly elastic body, then the application of the *div* operator leads to the vector V^{ν} with the components

$$V^0 = (-tH - X^J \rho v_j F^j{}_J - x^j \rho_o v_j)c_o, \tag{45}$$

$$V^I = tP^I{}_j v^j - X^J b^I{}_J + x^j P^I{}_j,$$

and to the conservation law

$$\frac{\partial V^0}{\partial \hat{t}} + \frac{\partial V^I}{\partial X^I} = 0 \tag{46}$$

or with (17)

$$\hat{t} T^{\nu}{}_{0,\nu} + X^I T^{\nu}{}_{I,\nu} = 0 \tag{47}$$

which states in words that

> $\hat{t}$ times conservation of energy plus X^I times conservation of material momentum equals a divergence-free expression.

After transforming Eq. (4.87) in Maugin (1993) to the notation used here, Eqs. (46) and (4.87) are identical. Maugin's statement (4.88)

> X^J times conservation of material momentum plus t times conservation of energy plus x^i times equation of motion equals a divergence-free expression

coincides with our statement above, since here it has been assumed from the beginning, that conservation laws exist only along solutions of $E_j(L) = 0$, i.e., the equations of motion are satisfied identically. Maugin's derivation, however, is quite different from our approach.

Finally, the application of the *curl* operator does not lead to a conservation law because $T_{\nu\mu}$ is not symmetric, i.e., $T_{\nu\mu} \neq T_{\mu\nu}$.

Considering $\mu = 0$ and $\nu = J$ in Eq. (21), in the absence of inhomogeniety forces $j_\nu = 0$ and volume forces $f_i = 0 (V = 0)$, we find

$$(c_o t T^{\lambda}{}_J)_{,\lambda} - (X_J T^{\lambda}{}_o)_{,\lambda} = T^0{}_J + c_o t\left[\frac{\partial T^0{}_J}{\partial \hat{t}} + \frac{\partial T^I{}_J}{\partial X^I}\right] - X_J\left(\frac{\partial T^0{}_0}{\partial \hat{t}} + \frac{\partial T^I{}_0}{\partial X^I}\right) - T^0{}_J = -c_o \rho v_j F^j{}_J - \frac{1}{c_o} v_j P^j{}_J$$
$$= -(\text{field intensity} + \text{field momentum density}). \tag{48}$$

Thus rotation in space and time results in a balance rather than a conservation law.

Considering $\mu = I$ and $\nu = J$ the general equation (21) yields

$$(X_I T^{\lambda}{}_J - X_J T^{\lambda}{}_I)_{,\lambda} = T_{IJ} - T_{JI}. \tag{49}$$

This right-hand side can be rearranged as a divergence for isotropic materials leading to (Kienzler and Herrmann, 2000)

$$\frac{\partial}{\partial t}(\varepsilon^{IKJ}X_K\rho_o v_j F^j{}_J - \varepsilon^{ikj}F^I{}_i x_k \rho_o v_j) + \frac{\partial}{\partial X^N}(\varepsilon^{IKJ}X_K b^N{}_J + \varepsilon^{ikj}F^I{}_i\, x_k P^N{}_j) = 0. \tag{50}$$

Thus rotation about the time axis ($\hat{t}$ fixed) leads to a conservation law, provided the material is isotropic.

3.4. Example

As a specific example we consider a linearly elastic bar with Young's modulus E, cross-section A, density ρ_o and displacement u. The prime indicates differentiation with respect to the axial coordinate X and a dot indicates differentiation with respect to time. The kinetic energy T and the strain energy W are, with $\mu = \rho_o A$, mass per unit of length

$$T = \frac{1}{2}\mu\dot{u}^2, \qquad W = \frac{1}{2}EAu'^2. \tag{51}$$

The four components of T^μ_ν are

$$\begin{bmatrix} T^0{}_0 & T^0{}_1 \\ T^1{}_0 & T^1{}_1 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{2}\mu\dot{u}^2 + \frac{1}{2}EAu'^2\right) & -Pu'c_o \\ +N\dot{u}\frac{1}{c_o} & +\frac{1}{2}\left(\mu\dot{u}^2 + \frac{1}{2}EAu'^2\right) \end{bmatrix} = \begin{bmatrix} -H & -c_oR \\ S\frac{1}{c_o} & B \end{bmatrix} \tag{52}$$

with the notation

$$P = \mu\dot{u} \quad \text{Physical momentum,} \tag{53}$$

$$N = EAu' \quad \text{Normal force,} \tag{54}$$

$$H = \tfrac{1}{2}(\mu\dot{u}^2 + EAu'^2) \quad \text{Hamiltonian or total energy,} \tag{55}$$

$$R = Pu' \quad \text{Field momentum or material momentum density,} \tag{56}$$

$$S = N\dot{u} \quad \text{Field intensity or energy flux,} \tag{57}$$

$$B = \tfrac{1}{2}(\mu\dot{u}^2 + EAu'^2) = L + Nu' \quad \text{Material momentum.} \tag{58}$$

It is interesting to note that the Hamiltonian and the material momentum are identical in this one-dimensional space.

The derivatives $T^\mu{}_{\nu,\mu} = 0$ lead to two conservation laws:

$$\frac{\partial}{\partial\hat{t}}T^0{}_0 + \frac{\partial}{\partial X}(T^1{}_0) = -\frac{1}{c_o}[H^\bullet - (N\dot{u})'] = 0 \tag{59}$$

which is a statement of conservation of energy and

$$\frac{\partial}{\partial\hat{t}}T^0{}_1 + \frac{\partial}{\partial X}T^1{}_1 = -[(Pu')^\bullet - B'] \tag{60}$$

which is a statement of conservation of material momentum.

The application of the *div* operator leads first to

$$tH + XR = -V^0 \tag{61}$$

which may be labelled the energy–field momentum virial (scalar moment) and to

$$tS + XB = V^1 \tag{62}$$

which is the material momentum–energy flux (field intensity) virial.

The final result is

$$\frac{\partial V^0}{\partial t} + \frac{\partial V^1}{\partial X} = 0 \qquad (63)$$

which is a conservation law, stating that the time rate of change of the virial density V^0 is balanced by the spatial rate of change of the flux V^1.

Turning now our attention to the application of the *curl* operator we calculate first

$$tc_oR - XH = -C^0 \qquad (64)$$

which is the energy–field momentum *curl*, and then

$$c_otB - XS = C^1 \qquad (65)$$

which is the material momentum–energy flux *curl*.

The final result is

$$\frac{\partial C^0}{\partial t} + \frac{\partial C^1}{\partial X} = -(c_o^2R + S) \qquad (66)$$

which is not a conservation but a balance law stating that the time rate of change of C^0 and the spatial rate of change of the flux C^1 are equal to a source term which is the negative of the sum of material momentum density and the energy flux.

4. Elastodynamics in Eulerian description

4.1. Lagrangian function and the energy–momentum tensor

In this description the role of the dependent and independent variables is reversed, as compared to the Lagrangian description above, i.e.,

$$\mathscr{L} = \mathscr{L}(c_ot, x^j; X^J, X^J{}_{,v}). \qquad (67)$$

We introduce a different symbol $\mathscr{L}$ for the Lagrangian function in Eulerian description to indicate, firstly that the functional dependence is different from L and, secondly, that the Lagrangian $\mathscr{L}$ is the action per unit of volume of the actual configuration whereas L is the action per unit of volume of the reference configuration. Thus, with (28), the following relation is valid

$$\mathscr{L} = J_FL. \qquad (68)$$

By contrast to the Lagrangian description, the "tensor" $\mathscr{T}^\mu{}_v$ (corresponding to $T^\mu{}_v$ in the Lagrangian description) cannot be written down in a straight forward manner, but is rather the result of a certain power-law expansion of relevant quantities. Since a detailed derivation may be found in Kienzler and Herrmann (2003) it will not be repeated here. We merely recall that the Euler–Lagrange equations now represent conservation (or balance) of material momentum (Eshelby) and the $\mathscr{T}^\mu{}_v$ "tensor" represents the mass-stress "tensor", which may be written out as

$$\mathscr{T}^{\mu v} = \begin{bmatrix} \mathscr{T}^{00} & \mathscr{T}^{0j} \\ \mathscr{T}^{i0} & \mathscr{T}^{ij} \end{bmatrix} = \begin{bmatrix} \rho c_o^2 & \rho v^j c_o \\ \rho v^i c_o & \rho v^i v^j - \sigma^{ij} \end{bmatrix}, \qquad (69)$$

where v^i is not $\frac{\partial X^J}{\partial t}|_{x^i\text{fixed}} = V^J$ but $\frac{\partial x^j}{\partial t}|_{X^I\text{fixed}}$ as in the previous description. The connection between v^j and V^J is (cf., e.g., Maugin, 1993)

$$v^j = -F^j{}_JV^J. \qquad (70)$$

Instead of the mass density in the reference configuration, ρ_o, the mass density of the current configuration, ρ, is used with the interrelation

$$\rho_o = J_F \rho. \tag{71}$$

The connection between the first Piola–Kirchhoff stress $P^I{}_j$ of the Lagrangian description and the Cauchy stress $\sigma^i{}_j$ is

$$\sigma^i{}_j = J_F^{-1} F^i{}_I P^I{}_j, \tag{72}$$

where J_F^{-1} is the Jacobian of the inverse transformation $X^K = \chi^{K^{-1}}(c_o t, x^k)$ (cf. Maugin, 1993). Differentiation of $\mathscr{T}^{\mu\nu}$ leads now first to

$$\mathscr{T}^{\nu 0}{}_{,\nu} = 0 = c_o\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v^j)\right) \tag{73}$$

which represents conservation of mass and next to

$$\mathscr{T}^{\nu j}{}_{,\nu} = \frac{\partial}{\partial t}(\rho v^j) + \frac{\partial}{\partial x^i}(\rho v^i v^j - \sigma^{ij}) = 0 \tag{74}$$

which represents conservation of physical momentum, i.e., the equations of motions, which can be written as

$$\rho \frac{Dv^j}{Dt} - \frac{\partial \sigma^{ij}}{\partial x^i} = 0 \tag{75}$$

with

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + v^i \frac{\partial()}{\partial x^i} \tag{76}$$

provided mass is conserved.

4.2. Applications of div and curl

Since the associated Lagrangian function is not homogeneous of any degree, the operation of *div* does not lead to a conservation law, but rather to the balance law

$$\frac{\partial}{\partial t}(x_j \rho v^j + c_o^2 t \rho) + \frac{\partial}{\partial x^i}[x_j(\rho v^i v^j - \sigma^{ij}) + c_o^2 t \rho v^i] = \rho c_o^2 + (\rho v^i v_i - \sigma^i{}_i) = \mathscr{T}^\nu{}_\nu. \tag{77}$$

The source term is equal to the trace of the mass-stress "tensor".

The "tensor" $\mathscr{T}^{\mu\nu}$ is, however, symmetric which indicates that the operation of *curl* will lead to a conservation law of the form

$$\frac{\partial}{\partial t}[t\rho v^j - x^j \rho] + \frac{\partial}{\partial x^i}[t(\rho v^i v^j - \sigma^{ij}) - x^j \rho v^i] = 0, \tag{78}$$

or expressed in words

t times equation of motion minus x_j times conservation of mass equals a divergence-free expression.

4.3. Example

As an illustration we consider again the linearly elastic bar employing the same notation as above. The "tensor" $\mathscr{T}^{\mu\nu}$ has the components

$$\mathscr{T}^{\mu\nu} = \begin{bmatrix} \mathscr{T}^{00} & \mathscr{T}^{01} \\ \mathscr{T}^{10} & \mathscr{T}^{11} \end{bmatrix} = \begin{bmatrix} \mu c_o^2 & \mu \dot{u} c_o = P c_o \\ \mu \dot{u} c_o = P c_o & \mu \dot{u}^2 - N = K \end{bmatrix}. \tag{79}$$

Differentiation leads first to

$$\frac{\partial}{\partial \hat{t}}(\mathscr{T}^{00}) + \frac{\partial}{\partial x}(\mathscr{T}^{10}) = c_o[\dot{\mu} + (\mu\dot{u})'] = 0 \quad \text{or} \quad \frac{\partial \mu}{\partial t} + \frac{\partial P}{\partial x} = 0, \tag{80}$$

and expresses conservation of mass, and second to

$$\frac{\partial}{\partial \hat{t}}(\mathscr{T}^{01}) + \frac{\partial}{\partial x}(\mathscr{T}^{11}) = (\mu\dot{u})^{\bullet} + (\mu\dot{u}^2 - N)' = 0 \quad \text{or} \quad \frac{\partial P}{\partial t} + \frac{\partial K}{\partial x} = 0. \tag{81}$$

Here P is the physical momentum as before and K may be called a kinetic force.

The right-hand side of (81) may be rewritten as

$$\dot{u}[\dot{\mu} + (\mu\dot{u})'] + \mu(\ddot{u} + \dot{u}u'^{\bullet}) - N' = 0. \tag{82}$$

The first term expresses again mass conservation and the parentheses of the second may be rewritten with (76) as a material time derivative

$$\ddot{u} + \dot{u}u'^{\bullet} = \frac{D\dot{u}}{Dt} \tag{83}$$

resulting in the standard form of the equation of motion of the elastic bar

$$\mu \frac{D\dot{u}}{Dt} = N'. \tag{84}$$

Application of *div* leads first to

$$c_o^2 t\mu + xP = V^0 \tag{85}$$

which is the virial of the mass-physical momentum, and then to

$$c_o^2 tP + xK = V^1 \tag{86}$$

which is the physical momentum–kinetic force virial.

The result of a further calculation is

$$\frac{\partial V^0}{\partial t} + \frac{\partial V^1}{\partial x} = \mu c_o^2 + K, \tag{87}$$

which states that the sum of the time rate of change of the virial density V^0 and the flux of the virial intensity V^1 are balanced by a source term which is the sum of the density μc_o^2 and the kinetic force K. Thus not a conservation, but a balance law results.

By contrast, application of *curl* does lead to a conservation law. First we calculate the mass-physical momentum *curl* C^0 as

$$-x\mu + tP = C^0 \tag{88}$$

and then the physical momentum-kinetic force *curl* C^1 as

$$-xP + tK = C^1, \tag{89}$$

and a further calculation leads to the result

$$\frac{\partial C^0}{\partial t} + \frac{\partial C^1}{\partial x} = 0 \tag{90}$$

Table 1
Juxtaposition of the results in Lagrangean and Eulerian descriptions

Elastodynamics in Lagrangian description		Elastodynamics in Eulerian description	
Euler–Lagrange eqs. $E_j(L) = 0$	Balance involving *phys.* momentum (equations of motion)	Euler–Lagrange eqs. $E_J(\mathscr{L}) = 0$	Balance involving *mat.* momentum (Eshelby relation)
Gradient $\begin{cases} T^{v}{}_{0,v} = 0 \\ T^{v}{}_{J,v} = 0 \end{cases}$	Conservation of energy Balance involving *mat.* momentum (Eshelby relation)	Gradient $\begin{cases} \mathscr{T}^{v}{}_{0,v} = 0 \\ \mathscr{T}^{v}{}_{J,v} = 0 \end{cases}$	Conservation of mass Balance involving *phys.* momentum (equations of motion)
Divergence	Leads to a *conservation* law	Divergence	Leads to a *balance* law
Curl	Leads to a *balance* law	Curl	Leads to a *conservation* law

which states that the time rate of change of the curl density C^0 is balanced by the flux of the curl intensity C^1.

5. Concluding remarks

The investigation presented in this paper has revealed new relationships existing in elastodynamics whose establishment was made possible by considering time on the same level as space, i.e., the point of view which is always adopted in the theory of relativity but is not common in continuum mechanics.

The principal results obtained are summarized in Table 1 which juxtaposes the two possible descriptions in elastodynamics, namely the Lagrangian and the Eulerian. Starting with a Lagrangian function, the Euler–Lagrange equations, as seen, lead, as should be expected, to different relations involving the physical momentum in the Lagrangian and the material momentum in the Eulerian description. Application of *grad*, *div* and *curl* leads in succession to different results, as is seen in the table.

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